Published by Institute of Physics Publishing for SISSA

Received: *August 15, 2005*
Revised: *September 14, 2005*
Accepted: *September 22, 2005*
Published: *October 10, 2005*

Notes on the S-matrix of bosonic and topological non-critical strings

Shin Nakamura and Vasilis Niarchos

The Niels Bohr Institute
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
E-mail: nakamura@nbi.dk, niarchos@nbi.dk

Abstract: We show that the equivalence between the $c = 1$ non-critical bosonic string and the $\mathcal{N} = 2$ topologically twisted coset SL(2)/U(1) at level one can be checked very naturally on the level of tree-level scattering amplitudes with the use of the Stoyanovsky-Ribault-Teschner map, which recasts H_3^+ correlation functions in terms of Liouville field theory amplitudes. This observation can be applied equally well to the topologically twisted $\mathrm{SL}(2)_n/\mathrm{U}(1)$ coset with $n > 1$, which has been argued recently to be equivalent with a $c < 1$ non-critical bosonic string whose matter part is defined by a time-like linear dilaton CFT.

Keywords: Superstrings and Heterotic Strings, Topological Strings, Bosonic Strings.

http://jhep.sissa.it/archive/papers/jhep102005025/jhep102005025.pdf

JHEP10(2005)025

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1. Introduction

String theories in two dimensions [1−3] and $\mathcal{N} = 2$ topological strings [4, 5] have been studied over the years in much detail. In both cases the dynamics are governed by some integrable structure. Indeed, two dimensional non-critical strings have been re-formulated and solved in many cases in terms of a dual matrix model [1−3, 6−8]. The emergence of the matrix model has been understood recently in terms of a holographic open/closed string duality [9−12]. Similarly, $\mathcal{N} = 2$ topological strings on generic non-compact Calabi-Yau spaces have been argued to be dual to corresponding finite N or large N matrix models through a rather different class of holographic open/closed string dualities [13−16]. All these features are interesting and provide a useful framework, where many aspects of string theory and quantum gravity can be studied efficiently.

Because of the presence of a topological/integrable structure in both two-dimensional and topological string theories, it is perhaps not very surprising that we can find examples, where a direct connection exists between a two dimensional non-critical string theory and a topological string theory on some appropriate target space. An example of this sort of equivalence involves the $c = 1$ non-critical bosonic string theory at the self-dual radius and the topologically twisted $\mathcal{N} = 2$ coset CFT $\mathrm{SL}(2, \mathbb{R})_1/\,\mathrm{U}(1)$ at level one [17]. Strong evidence in favor of this equivalence has been presented on the level of BRST cohomologies and correlation functions in [17, 18]. It has been further argued that there is another equivalence between the topologically twisted coset $\mathrm{SL}(2,\mathbb{R})_1/\mathrm{U}(1)$ and the topological Landau-Ginzburg (LG) model with superpotential $W(X) = X^{-1}$ [19]. This suggests an additional connection between the topologically twisted coset CFT and the topological string theory on the conifold [20, 21]. The existence of the same set of deformations on both sides of this equivalence have led to the intriguing claim that the $\mathcal{N} = 2$ topological string on generic Calabi-Yau threefold singularities is described by appropriate deformations of the $c = 1$ non-critical bosonic string theory [22].

One may wonder whether it is possible to find bosonic string equivalents to the topologically twisted coset $\mathrm{SL}(2)_n/\mathrm{U}(1)$ at general values of the level n. It turns out that the answer to this question is affirmative. The corresponding bosonic string has been determined recently in [23]. For integer levels n the precise statement is that the A-model topological string on the $\mathbb{Z}_n$ orbifold $(\frac{\mathrm{SL}(2,\mathbb{R})_n}{\mathrm{U}(1)})/\mathbb{Z}_n$ is equivalent to the non-minimal $c = 1-6(n-1)^2/n$ non-critical bosonic string. The definition of the non-minimal $c = 1 - 6(n-1)^2/n$ bosonic string involves the standard Liouville sector with linear dilaton slope $Q = \frac{1}{\sqrt{n}}(n+1)$ plus a time-like boson X_0 that is compactified at radius $R_{X_0} = \sqrt{2n}$ and has linear dilaton slope $q = \frac{1}{\sqrt{n}}(n-1)$. As expected, for $n = 1$ the non-minimal string reduces to the usual $c = 1$ bosonic string and we recover the equivalence of the previous paragraph. Moreover, as in the case of the $c = 1$ string, these theories are expected to be equivalent to topological LG models with a more general superpotential $W(X) = -\mu X^{-n}$. In [23] Takayanagi supported the claim for this general n equivalence with an analysis of the respective BRST cohomologies, a computation of tree-level two- and three-point functions in the topologically twisted $\mathrm{SL}(2,\mathbb{R})_n/\mathrm{U}(1)$ coset and a computation of higher N-point functions using topological LG techniques.

In this note, we provide further evidence for this equivalence with a computation of a large class of tree-level N-point correlation functions directly in the A-model topological string on $(\frac{\mathrm{SL}(2,\mathbb{R})_n}{\mathrm{U}(1)})/\mathbb{Z}_n$. At first sight, this computation appears to be a formidable task that would require explicit knowledge of arbitrary N-point functions of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ theory. Fortunately, however, it turns out that we can establish the agreement of the sphere correlation functions on both sides of the equivalence in a rather simple and straightforward way. This new non-trivial check relies heavily on work by Ribault and Teschner that appeared recently in two beautiful papers [24, 25]. In the first of these papers, an intriguing new dictionary was established between tree-level N-point functions in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model at level n and $(2N-2)$-point functions in Liouville field theory. This correspondence is based on a map between solutions of the Belavin-Polyakov-Zamolodchikov (BPZ) and Knizhnik-Zamolodchikov (KZ) systems of partial differential equations that was discovered originally by Stoyanovsky in [26]. In a second paper [25] this dictionary was extended to include also spectral-flow violating amplitudes in $\mathrm{SL}(2,\mathbb{R})$ (see also related work in [27][1]). The $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model is the Euclidean version of the $\mathrm{SL}(2,\mathbb{R})$ WZW model and we will go from one to the other by analytic continuation. Partial justification for the validity of this continuation will be given by the consistency of our results.

[1] In [27] one can also find a nice discussion on the implications of the Stoyanovsky-Ribault-Teschner map in Little String Theory.

To summarize, our strategy is the following. We begin with a certain class of sphere N-point correlation functions in the topologically twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ theory. Then, we show that these correlation functions can be reduced easily to a set of $\mathrm{SL}(2,\mathbb{R})$ amplitudes with maximal spectral-flow number violation and we make use of the Stoyanovsky-Ribault-Teschner (SRT) map to recast these amplitudes in terms of Liouville field theory N-point functions. Through this map one can see the non-minimal bosonic string amplitudes emerging naturally with the right parameters and with precisely those insertions expected from the bosonic/topological correspondence of [23]. One of the most appealing features of this approach is that it gives a very natural and straightforward way to go from the topological string to the non-critical bosonic string. We believe that similar techniques will be useful in the study of topological strings in many other cases where the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset is involved. This includes topological strings in generalized conifold singularities in CY threefolds and the $\mathcal{N}=2$ string (equivalently the $\mathcal{N}=4$ topological string) in the vicinity of K3 singularities. We make several comments on such generalizations in the final section.

The organization of this paper is as follows. In section 2 we review the basic features of the equivalence proposed in [23] and set our notation straight. In section 3 we present the SRT map, which will be used heavily in section 4 to compute topological string N-point functions. We conclude in section 5 with a brief discussion of some interesting open problems.

2. Topologically twisted 2D black holes and bosonic non-critical strings

In this section we fix the notation and state in precise terms the conjectured equivalence [23] between the $c \leq 1$ non-minimal strings and the A-model topological string on $(\frac{\mathrm{SL}(2,\mathbb{R})_n}{\mathrm{U}(1)})/\mathbb{Z}_n$. We follow the notation of [23] and set $\alpha' = 2$ throughout the rest of this note.

2.1 The topologically twisted coset

The $\mathcal{N}=2$ supersymmetric $\mathrm{SL}(2,\mathbb{R})_n/\mathrm{U}(1)$ model at level $n > 0$ is a two-dimensional CFT with central charge $c = 3 + 6/n$. It can be obtained by gauging an appropriate $\mathrm{U}(1)$ in the product of a free Dirac fermion $(\psi, \psi^\dagger)$ and the *bosonic* $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model at level $n+2$. We summarize briefly a few of the relevant details to fix the notation.

The bosonic $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model involves an $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ current algebra, whose left-moving part is captured by the current OPE's

$$J^3(z)J^3(0) \sim -\frac{n+2}{2z^2}\,, \qquad J^3(z)J^\pm(0) \sim \pm\frac{J^\pm(0)}{z}\,, \qquad J^+(z)J^-(0) \sim \frac{n+2}{z^2} - \frac{2J^3(0)}{z}\,. \tag{2.1}$$

A similar set of OPE's determines the right-moving current algebra $\mathrm{SL}(2,\mathbb{R})_R$. Now consider the product CFT of this $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model with a free CFT consisting of a complex fermion $(\psi, \psi^\dagger)$. As usual, the free OPE of the complex fermion ψ is $\psi(z)\psi^\dagger(0) \sim 1/z$ and for later convenience we bosonize the fermion with a boson H, so that

$$\psi = e^{iH}\,, \qquad \psi\psi^\dagger = i\partial H\,. \tag{2.2}$$

The U(1) current, whose gauging defines the $\mathcal{N}=2$ supersymmetric coset, takes the form

$$J_g = J^3 - \psi^\dagger\psi - i\sqrt{\frac{n}{2}}\partial X\,, \tag{2.3}$$

where X is a free U(1) boson with the standard OPE $X(z)X(0) \sim -\log z$. To perform the U(1) gauging on the BRST level, we can add a system of $c=-2$ ghosts (ξ,η) and define the BRST charge

$$Q_{\mathrm{U}(1)} = \int dz\; \xi(z)J_g(z)\,. \tag{2.4}$$

The result of this procedure is a theory with $\mathcal{N}=(2,2)$ worldsheet supersymmetry. Geometrically, the target space of this theory looks like a cigar (or the Euclidean version of a 2D black hole) with asymptotic radius $R=\sqrt{2n}$. The left-moving part of the $\mathcal{N}=2$ superconformal current algebra takes the form

$$\begin{aligned} J_R &= \frac{n+2}{n}\psi^\dagger\psi - \frac{2}{n}J^3 \simeq -i\partial H - i\sqrt{\frac{2}{n}}\partial X\,,\\ G^+ &= \sqrt{\frac{2}{n}}\psi^\dagger J^+\,, \qquad G^- = \sqrt{\frac{2}{n}}\psi J^-\,. \end{aligned} \tag{2.5}$$

To obtain the second expression for the $\mathrm{U}(1)_R$ current we used the U(1) gauging condition. Moreover, for later purposes, it will be convenient to define an equivalent $\mathrm{U}(1)_R$ current

$$J_R' = J_R - 2J_g = -3i\partial H - 2J^3 + i\sqrt{\frac{2}{n}}(n-1)\partial X\,. \tag{2.6}$$

Of course, similar statements apply also to the right-movers, but with a few minor sign differences, which we summarize here[2]

$$\begin{aligned} \bar J_g &= -\bar J^3 - \bar\psi\bar\psi^\dagger - i\sqrt{\frac{n}{2}}\bar\partial X\,,\\ \bar J_R &= \frac{n+2}{n}\bar\psi\bar\psi^\dagger + \frac{2}{n}\bar J^3 \sim \bar\psi\bar\psi^\dagger - i\sqrt{\frac{2}{n}}\bar\partial X\,,\\ \bar J_R' &= \bar J_R - 2\bar J_g = 3\bar\psi\bar\psi^\dagger + 2\bar J^3 + i\sqrt{\frac{2}{n}}(n-1)\bar\partial X\,. \end{aligned} \tag{2.7}$$

Certain primary fields of the supersymmetric coset will soon play an important role. They can be constructed from the bosonic $\mathrm{SL}(2,\mathbb{R})_{n+2}$ primary fields, which we now quickly review. The bosonic $\mathrm{SL}(2,\mathbb{R})_{n+2}$ model possesses a set of affine primary fields $\Phi_{j,m,\bar m}$ that can be expressed as

$$\Phi_{j,m,\bar m} = V_{j,m,\bar m}e^{\sqrt{\frac{2}{n+2}}(mX_3+\bar m\bar X_3)}\,. \tag{2.8}$$

In this relation, X_3 is a canonically normalized boson that has been defined so that

$$J^3(z) = -\sqrt{\frac{n+2}{2}}\partial X_3\,. \tag{2.9}$$

Also, $V_{j,m,\bar m}$ is a primary field of the bosonic coset $\mathrm{SL}(2,\mathbb{R})_{n+2}/\,\mathrm{U}(1)$ that carries by defi-

[2]We use the following notation. Bars $^-$ will denote the right-movers and daggers † will denote complex conjugation on the target fields.

nition no J_0^3 charge. The conformal dimensions of these fields are

$$\Delta(\Phi_{j,m}) = -\frac{j(j+1)}{n}\,, \qquad \Delta(V_{j,m}) = -\frac{j(j+1)}{n} + \frac{m^2}{n+2}\,. \tag{2.10}$$

In these expressions the quantum numbers j and m can take different values depending on the type of SL(2) representation we want to consider. For the lowest weight discrete representations $\mathcal{D}_j^+$ j is real and $m = -j, -j+1, -j+2, \ldots$. For the highest weight discrete representations $\mathcal{D}_j^-$ j is again real, but $m = j, j-1, j-2, \ldots$. Finally, for the continuous representations $j = -\frac{1}{2} + is$, $s \in \mathbb{R}$, but these representations will not appear in the topologically twisted theory.

In addition, the spectrum of the bosonic $SL(2,\mathbb{R})$ WZW model includes representations that arise from the spectral flow operation [28], some details of which are summarized in the appendix. The spectral flow of the primary fields $\Phi_{j,m,\bar{m}}$ by an amount w will be denoted as $\Phi^w_{j,m,\bar{m}}$ whose explicit form is

$$\Phi^w_{j,m,\bar{m}} = e^{\sqrt{\frac{2}{n+2}}\left[(m+\frac{n+2}{2}w)X_3 + (\bar{m}+\frac{n+2}{2}w)\bar{X}_3\right]} V_{j,m,\bar{m}}\,. \tag{2.11}$$

We will see in a moment that the spectral flow operation appears naturally as part of the topological twist.

In any case, given the above primary fields one can easily construct primary fields of the supersymmetric coset by simply demanding that the total U(1)-gauging charge Q_g is zero. For example, one can check that the primary fields

$$e^{i(sH - \bar{s}\bar{H})}\Phi_{j,m,\bar{m}} e^{i\sqrt{\frac{2}{n}}\left[(s+m)X - (\bar{s}+\bar{m})\bar{X}\right]} \tag{2.12}$$

satisfy this condition and are therefore allowed primary fields of the supersymmetric coset.

We are now ready to consider the topologically twisted theory. We define the A-model topological twist of the $\mathcal{N}=2$ coset by the usual twist of the worldsheet stress tensor[3]

$$T \to T + \frac{1}{2}\partial J'_R\,, \qquad \bar{T} \to \bar{T} - \frac{1}{2}\bar{\partial}\bar{J}'_R\,. \tag{2.13}$$

Notice that, as in [23], we choose to define the topological twist with the use of the gauge equivalent $U(1)_R$ current J'_R. One can check that with these conventions the BRST cohomology of the topologically twisted theory will consist of (c,a) primaries of the $\mathcal{N}=2$ coset CFT. By definition, the (c,a) primary fields are NS-sector primary fields $\mathcal{O}_{NS}$ satisfying the constraints

$$\oint dz\; G^+(z)\cdot\mathcal{O}_{NS}(0) = 0\,, \qquad \oint d\bar{z}\; \bar{G}^-(\bar{z})\cdot\mathcal{O}_{NS}(0) = 0\,. \tag{2.14}$$

Some of these primaries (in fact the ones that will be relevant for this paper) can be determined simply by setting $s = \bar{s} = 0$, $m = \bar{m} = j$ in (2.12). For later convenience, we denote these fields as

$$V^{NS}_{-\frac{2j+1}{2n}} = \Phi_{j,j,j} e^{i\sqrt{\frac{2}{n}}j(X-\bar{X})}\,. \tag{2.15}$$

They have scaling dimensions $\Delta = \frac{q_R}{2} = -\frac{j}{n}$, $\bar{\Delta} = -\frac{\bar{q}_R}{2} = -\frac{j}{n}$ and vanishing momentum in the angular direction of the cigar.

[3]It is not difficult to consider also the B-model topological twist, but this will not be done explicitly in this paper.

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An equivalent representation of the (c, a) primary fields can be given in the Ramond sector as R-sector ground states [29]. The corresponding vertex operators will be denoted as $V^R_{-\frac{2j+1}{2n}}$. They can be obtained from the NS-sector vertex operators (2.15) by $(-1/2, 1/2)$ $\mathcal{N}=2$ spectral flow (the relevant conventions are summarized in the appendix)

$$V^R_{-\frac{2j+1}{2n}} = e^{-\frac{i}{2}\sqrt{\frac{n+2}{n}}(-X_R+\bar{X}_R)} V^{NS}_{-\frac{2j+1}{2n}} = e^{\frac{i}{2}(H+\bar{H})} \Phi_{j,j,j} e^{i\sqrt{\frac{2}{n}}(j+\frac{1}{2})(X-\bar{X})} . \tag{2.16}$$

The canonically normalized boson X_R appearing in the first equality is defined so that

$$J_R(z) = -i\sqrt{\frac{n+2}{n}} \partial X_R(z) . \tag{2.17}$$

Going in the opposite direction, physical states of the topologically twisted coset in the NS sector can be obtained from the R-sector ground states by a $(\frac{1}{2}, -\frac{1}{2})$ $\mathcal{N}=2$ spectral flow transformation, but now according to (2.13) we define the $\mathcal{N}=2$ spectral flow transformation in terms of the modified boson X'_R. Applying this operation to the R-ground states (2.16) gives the vertex operators

$$V_{-\frac{2j+1}{2n}} = e^{-\frac{i}{2}\sqrt{\frac{n+2}{n}}(X'_R-\bar{X}'_R)} V^R_{-\frac{2j+1}{2n}} = e^{-i(H+\bar{H})} \Phi^{w=1}_{j,j,j} e^{i\sqrt{\frac{2}{n}}(j+\frac{n}{2})(X-\bar{X})} . \tag{2.18}$$

Note that with this prescription part of the topological twist is an $\mathrm{SL}(2,\mathbb{R})$ spectral flow operation, since X'_R contains a term proportional to X_3.

We conclude this subsection with a few additional comments on certain features of the topologically twisted coset theory:

(1) In what follows we consider a $\mathcal{Z}_n$ orbifold projection of the topologically twisted coset $\mathrm{SL}(2,\mathbb{R})_n/\mathrm{U}(1)$. For this it is necessary to have $n \in \mathbb{Z}_+$. The $\mathbb{Z}_n$ projection acts by translation $\frac{2\pi}{\sqrt{n/2}}$ on the boson X and reduces the asymptotic radius of the cigar from $R = \sqrt{2n} \to \sqrt{\frac{2}{n}}$ (in $\alpha' = 2$ units).

(2) The quantum numbers j in (2.18) are real and have been argued [23] to satisfy the constraint $2j+1 \in n\mathbb{Z}$. This is consistent with the above orbifold and can be deduced by demanding that the vertex operators $V_{-\frac{2j+1}{2n}}$ are mutually local with the SU(2) K^- operator. Further details can be found in [23].

(3) In standard discussions of the $\mathrm{SL}(2,\mathbb{R})$ WZW model or its U(1) coset, one demands that normalizable representations respect the unitarity bound $-(n+1)/2 < j < -1/2$. This bound will be dropped in the topologically twisted theory, since we focus mostly on non-normalizable representations.

(4) The topologically twisted coset exhibits many more physical states, which can be formulated most easily in the Wakimoto free field representation of $\mathrm{SL}(2,\mathbb{R})$ [17, 23]. For each of these states there is a corresponding state in the dual bosonic string. We described some of these states here corresponding to ghost number 1 tachyons, but there are more at this ghost number and ghost numbers 0 or 2 and arrange themselves

in SU(2) multiplets. An extensive list appears in [17, 23]. In this paper we focus on the physical states (2.18), since they have a simple formulation without any reference to the Wakimoto representation.

(5) It is well-known that the $\mathcal{N}=2$ coset $\mathrm{SL}(2)_n/\mathrm{U}(1)$ is mirror dual to the $\mathcal{N}=2$ Liouville theory [30]. It would be interesting to consider the topologically twisted $\mathcal{N}=2$ Liouville theory and repeat the analysis of [17, 23] directly in this context to obtain the non-trivial BRST cohomology of the $c \leq 1$ non-minimal bosonic string. This seems to be complicated by the absence of an obvious analogue of the Wakimoto representation.

2.2 The non-minimal $c \leq 1$ bosonic string

The non-minimal $c \leq 1$ bosonic string was first introduced in [31]. On the worldsheet level, this theory consists of a time-like linear dilaton theory with slope $q = \frac{1}{b} - b$ and the standard space-like Liouville theory with linear dilaton slope $Q = b + \frac{1}{b}$. In our conventions, the worldsheet action takes the form

$$\mathcal{S} = \frac{1}{4\pi} \int d^2z \left(-\partial X_0 \partial X_0 + \frac{q}{2\sqrt{2}} X_0 \mathcal{R}^{(2)} + \partial\phi\partial\phi + \frac{Q}{2\sqrt{2}} \phi \mathcal{R}^{(2)} + 2\pi\mu_L e^{\sqrt{2}b\phi} \right) \tag{2.19}$$

and the total central charge is

$$c_{tot} = c_{X_0} + c_\phi = 1 - 6q^2 + 1 + 6Q^2 = 26\,. \tag{2.20}$$

This theory is a natural extension of the $c = 1$ non-critical string, where $b = 1$ and the time-like boson has vanishing linear dilaton slope q. In general, the parameter b is restricted to lie within the range $0 \leq b \leq 1$. The second condition comes from the Seiberg bound on the Liouville potential $e^{\sqrt{2}b\phi}$.

The primary fields of the non-minimal bosonic string are

$$\mathcal{V}_{\alpha,\beta} = e^{\sqrt{2}(\alpha X_0 + \beta\phi)} \tag{2.21}$$

and have scaling dimensions

$$\Delta(\mathcal{V}_{\alpha,\beta}) = -\alpha(q-\alpha) + \beta(Q-\beta)\,. \tag{2.22}$$

Notice that the string coupling is

$$g_s = e^{\frac{1}{\sqrt{2}}(qX_0 + Q\phi)} \tag{2.23}$$

and the system will get strongly coupled at late times, but an appropriate Lorentz transformation [31] can be used to recast this theory as an ordinary linear dilaton vacuum perturbed by a time-dependent Liouville potential. A matrix model formulation of this theory has been discussed in [31].

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2.3 The correspondence

For integer numbers n it has been argued [23] that the non-minimal $c = 1 - 6(n-1)^2/n$ string at radius $R_{X_0} = \sqrt{2n}$ is equivalent to the topologically twisted A-model orbifold $(\frac{\mathrm{SL}(2,\mathbb{R})_{n+2}}{\mathrm{U}(1)})/\mathbb{Z}_n$. The explicit mapping of states can be found in [23]. In particular, the vertex operators that map to (2.18) are the bosonic string tachyons

$$c\bar{c}\ \mathcal{Y}_s = c\bar{c}\ \mathcal{V}_{\alpha_s - \frac{1}{\sqrt{n}}, \alpha_s}\,, \tag{2.24}$$

where c is the usual c ghost of the bosonic string, α_s is

$$\alpha_s = \frac{1}{\sqrt{n}}\left(\frac{1}{2} - ns\right) + \frac{\sqrt{n}}{2} \tag{2.25}$$

and the relation between s and the quantum number j appearing in (2.18) is $s = -\frac{2j+1}{2n}$.

Here it is perhaps useful to recall this mapping of states in the more familiar $c = 1$ context using the more standard notation of [32]. The topological string vertex operators V_s $(s = -j - \frac{1}{2} \in \frac{\mathbb{Z}}{2})$ in (2.18) correspond in the $c = 1$ bosonic string to the ghost number one tachyons

$$\begin{aligned} Y^{+}_{s,-s} &= c\bar{c}\ e^{-i\sqrt{2}sX_0} e^{\sqrt{2}(1-s)\phi}\,, \qquad s \geq 0\,, \\ Y^{-}_{s,s} &= c\bar{c}\ e^{i\sqrt{2}sX_0} e^{\sqrt{2}(1+s)\phi}\,, \qquad s \geq 0\,. \end{aligned} \tag{2.26}$$

The two indices $(s, \pm s)$ are $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ labels. In addition, there is a conjugate class of topological string vertex operators $\tilde{V}_s$, which will not be discussed explicitly here, that correspond to the $c = 1$ bosonic string tachyons

$$\begin{aligned} Y^{+}_{s,s} &= c\bar{c}\ e^{i\sqrt{2}sX_0} e^{\sqrt{2}(1-s)\phi}\,, \qquad s \geq 0\,, \\ Y^{-}_{s,-s} &= c\bar{c}\ e^{-i\sqrt{2}sX_0} e^{\sqrt{2}(1+s)\phi}\,, \qquad s \geq 0\,. \end{aligned} \tag{2.27}$$

The vertex operators $Y^{+}_{s,\pm s}$ appear also in [33] in another notation

$$T_k = c\bar{c}\ e^{ikX_0} e^{(-|k|+\sqrt{2})\phi} \tag{2.28}$$

with $k = -\sqrt{2}s$, $s \in \frac{\mathbb{Z}}{2}$. The absolute value $|k|$ in the exponent is chosen in order to satisfy the Seiberg bound. N-point functions on the sphere of these vertex operators have been computed in [34–36, 17, 23] using topological LG techniques and in [33, 31] using the worldsheet formulation of the bosonic string. In this paper we compute directly in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset.

3. The Stoyanovsky-Ribault-Teschner map

Recently the authors of [24] formulated a very precise map between sphere correlation functions in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model and correlation functions in Liouville field theory. The proof of this map is based on a relation between the KZ and BPZ systems of partial differential equations in the two theories and provides the extension of a similar observation by Stoyanovsky in [26] for SU(2) WZW models.

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The basic formula that was obtained in [24] provides a map between winding number conserving N-point functions in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model at level k and $(2N-2)$-point functions in Liouville field theory with linear dilaton slope $Q = b + b^{-1}$ and $b^2 = \frac{1}{k-2}$. More explicitly,

$$\begin{aligned}
&\left\langle \prod_{i=1}^{N} \Phi_{j_i,m_i,\bar{m}_i}(z_i,\bar{z}_i) \right\rangle = \\
&= \prod_{i=1}^{N} \mathcal{N}^{j_i}_{m_i,\bar{m}_i} \delta\left(\sum_{\ell=1}^{N} m_\ell\right) \delta\left(\sum_{\ell=1}^{N} \bar{m}_\ell\right) \times \\
&\quad \times \prod_{a=1}^{N-2} \int d^2 y_a \; \mathcal{F}_k(\{z_i,\bar{z}_i\};\{y_a,\bar{y}_a\}) \left\langle \mathcal{V}_{\alpha_i}(z_i,\bar{z}_i) \mathcal{V}_{-\frac{1}{2b}}(y_a,\bar{y}_a) \right\rangle ,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\mathcal{F}_k(\{z_i,\bar{z}_i\};\{y_a,\bar{y}_a\}) &= \frac{2\pi^3 b}{\pi^{2N}(N-2)!} \prod_{i<j\leq N} (z_i - z_j)^{m_i+m_j+\frac{k}{2}} (\bar{z}_i - \bar{z}_j)^{\bar{m}_i+\bar{m}_j+\frac{k}{2}} \times \\
&\quad \times \prod_{a<b\leq N-2} |y_a - y_b|^k \prod_{r=1}^{N} \prod_{c=1}^{N-2} (z_r - y_c)^{-m_r-\frac{k}{2}} (\bar{z}_r - \bar{y}_c)^{-\bar{m}_r-\frac{k}{2}}
\end{aligned} \tag{3.2}$$

and

$$\mathcal{N}^{j}_{m,\bar{m}} = \frac{\Gamma(-j+m)}{\Gamma(1+j-\bar{m})} . \tag{3.3}$$

Following the notation of the previous section, the Liouville field theory vertex operators $\mathcal{V}_\alpha$ reduce to the exponential fields $e^{\sqrt{2}\alpha\phi}$ in the classical limit $b \to 0$. The SRT map involves $N-2$ degenerate vertex operators $\mathcal{V}_{-\frac{1}{2b}}$ and N vertex operators with

$$\alpha_i = b j_i + b + \frac{1}{2b} , \quad 1 \leq i \leq N , \tag{3.4}$$

which are in one-to-one correspondence with the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ primary fields. In these expressions the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ quantum numbers $(j,m,\bar{m})$ are restricted to the set of values

$$j = -\frac{1}{2} + is , \qquad m = \frac{n+ip}{2} , \qquad \bar{m} = \frac{-n+ip}{2} , \qquad n \in \mathbb{Z} , \quad s,p \in \mathbb{R} . \tag{3.5}$$

The corresponding primary fields belong to the continuous series and cover the full physical spectrum of the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ model. In order to apply this map to $\mathrm{SL}(2,\mathbb{R})$ correlation functions we need to perform an analytic continuation, both in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model and in Liouville field theory. In general, this analytic continuation is believed to hold and gives sensible results as we see below.

In addition, as we summarized above, the physical spectrum of string theory on $\mathrm{SL}(2,\mathbb{R})$ also includes spectral flowed representations. Correlation functions involving such primary fields are not expected to obey the KZ equations, but there is a natural generalization of the KZ equations [25] that allows for the inclusion of spectral flow. Then, one can show that with the appropriate modifications the SRT map with spectral flowed primary fields takes the form [25][4]

$$
\begin{aligned}
&\left\langle \prod_{i=1}^{N} \Phi^{w_i}_{j_i,m_i,\bar{m}_i}(z_i,\bar{z}_i)\right\rangle_{\sum w = r\geq 0} = \\
&= \prod_{i=1}^{N} \mathcal{N}^{j_i}_{m_i,\bar{m}_i}\delta^{(2)}\left(\sum_{\ell=1}^{N} m_\ell + \frac{k}{2}r\right) \times \\
&\times \prod_{a=1}^{N-2-r} \int d^2y_a\ \tilde{\mathcal{F}}_k(\{z_i,\bar{z}_i\};\{y_a,\bar{y}_a\})\left\langle \mathcal{V}_{\alpha_i}(z_i,\bar{z}_i)\mathcal{V}_{-\frac{1}{2b}}(y_a,\bar{y}_a)\right\rangle,
\end{aligned} \tag{3.6}
$$

where c_k is a k-dependent constant and

$$
\begin{aligned}
&\tilde{\mathcal{F}}_k(\{z_i,\bar{z}_i\};\{y_a,\bar{y}_a\}) = \\
&= \frac{2\pi^{3-2N} b c_k^r}{(N-2-r)!} \prod_{i<j\leq N} (z_i - z_j)^{\beta_{ij}} (\bar{z}_i - \bar{z}_j)^{\bar{\beta}_{ij}} \times \\
&\times \prod_{a<b\leq N-2-r} |y_a - y_b|^k \prod_{r=1}^{N} \prod_{c=1}^{N-2-r} (z_r - y_c)^{-m_r - \frac{k}{2}} (\bar{z}_r - \bar{y}_c)^{-\bar{m}_r - \frac{k}{2}}
\end{aligned} \tag{3.7}
$$

with

$$
\begin{aligned}
\beta_{ij} &= \frac{k}{2} + m_i + m_j - \frac{k}{2} w_i w_j - w_i m_j - w_j m_i\,, \\
\bar{\beta}_{ij} &= \frac{k}{2} + \bar{m}_i + \bar{m}_j - \frac{k}{2} \bar{w}_i \bar{w}_j - \bar{w}_i \bar{m}_j - \bar{w}_j \bar{m}_i\,.
\end{aligned} \tag{3.8}
$$

As we have already seen in section 2, part of the topological twist involves spectral flow in $\mathrm{SL}(2,\mathbb{R})$. Hence, the map (3.6) will be directly relevant for the topological string computation of the next section.

4. N-point functions

This section contains the main observation of this paper. Using the SRT map, we compute N-point functions of the topological string vertex operators (2.18) and relate them to N-point functions in the corresponding non-minimal $c \leq 1$ bosonic string. We begin with a brief review of a well-known prescription that allows us to recast the topological string correlation functions as correlation functions in the untwisted CFT.

4.1 General comments

With each physical state $|\mathcal{O}_\alpha\rangle$ in a topologically twisted (A-model) theory one can associate a complete superfield

$$
\mathcal{O}_\alpha = \mathcal{O}^{(0)}_\alpha + \theta \mathcal{O}^{(1,0)}_\alpha + \bar{\theta}\mathcal{O}^{(0,1)}_\alpha + \theta\bar{\theta}\mathcal{O}^{(1,1)}_\alpha\,, \tag{4.1}
$$

[4]Compared to the conventions of [25] we have $m_{ours} = -m_{there}$, $\bar{m}_{ours} = -\bar{m}_{there}$ and $w_{ours} = -w_{there}$.

or one can write in the NS-sector

$$\mathcal{O}_\alpha^{(1,0)} = G^-_{-1/2} \cdot \mathcal{O}_\alpha^{(0)}\,, \quad \mathcal{O}_\alpha^{(0,1)} = \bar{G}^+_{-1/2} \cdot \mathcal{O}_\alpha^{(0)}\,, \quad \mathcal{O}_\alpha^{(1,1)} = G^-_{-1/2}\bar{G}^+_{-1/2} \cdot \mathcal{O}_\alpha^{(0)}\,. \tag{4.2}$$

Accordingly, there are four topological string observables associated to these components:

$$\mathcal{O}_\alpha^{(0)}\,, \quad \oint dz\ \mathcal{O}_\alpha^{(1,0)}\,, \quad \oint d\bar{z}\ \mathcal{O}_\alpha^{(0,1)}\,, \quad \int d^2z\ \mathcal{O}_\alpha^{(1,1)}\,. \tag{4.3}$$

For example, for the topological vertex operators $V_{-\frac{2j+1}{2n}}$ appearing in (2.18) we get

$$V^{(1,0)}_{-\frac{2j+1}{2n}} \equiv G^-_{-1/2} \cdot V_{-\frac{2j+1}{2n}} = 2\sqrt{\frac{2}{n}}j\ e^{-i\bar{H}}\,\Phi^{w=1}_{j,j-1,j}e^{i\sqrt{\frac{2}{n}}(j+\frac{n}{2})(X-\bar{X})}\,. \tag{4.4}$$

Analogous results apply also to the $(0,1)$ and $(1,1)$ forms of this vertex operator.

In general, correlation functions $\langle\prod_i \mathcal{O}^{(0)}_{\alpha_i}\rangle$ of the 0-form operators $\mathcal{O}^{(0)}_{\alpha_i}$ can be determined easily by using factorization in terms of the three-point function [34]

$$c_{\alpha_1\alpha_2\alpha_2} = \langle \mathcal{O}^{(0)}_{\alpha_1}\mathcal{O}^{(0)}_{\alpha_2}\mathcal{O}^{(0)}_{\alpha_3}\rangle\,. \tag{4.5}$$

In a similar fashion, one can show that the more general correlation function of the type[5]

$$\left\langle \prod_{i=1}^{N}\mathcal{O}^{(0)}_{\alpha_i}\prod_{\ell=1}^{M}\int d^2z_\ell\ \mathcal{O}^{(1,1)}_{\alpha_\ell}\right\rangle \tag{4.6}$$

can be recast in terms of the perturbed three-point function

$$c_{\alpha_1\alpha_2\alpha_3}(t) = \left\langle \mathcal{O}^{(0)}_{\alpha_1}\mathcal{O}^{(0)}_{\alpha_2}\mathcal{O}^{(0)}_{\alpha_3}e^{\sum_n t_n\int d^2z\ \mathcal{O}^{(1,1)}_{\alpha_n}}\right\rangle \tag{4.7}$$

or in terms of correlation functions of the form

$$\mathcal{F}_{\alpha_1,\dots,\alpha_N} \equiv \left\langle \mathcal{O}^{(0)}_{\alpha_1}\mathcal{O}^{(0)}_{\alpha_2}\mathcal{O}^{(0)}_{\alpha_3}\prod_{\ell=4}^{N}\int d^2z_\ell\mathcal{O}^{(1,1)}_{\alpha_\ell}\right\rangle\,. \tag{4.8}$$

It is precisely this last type of correlation functions that we want to determine for the vertex operators (2.18). This can be achieved in the untwisted $\mathcal{N}=2$ coset CFT in the following way.

The topologically twisted theory exhibits a $\mathrm{U}(1)_R$ anomaly (or a $\mathrm{U}(1)_R$ background charge) of -$\frac{c}{3}$ on the sphere. In the original untwisted $\mathcal{N}=2$ superconformal theory, the $\mathrm{U}(1)_R$ current is not anomalous and the corresponding charge is conserved. Therefore, in order to compute the correlation function (4.8) in the untwisted theory we need to insert the background charge by hand. This can be achieved in many different ways, but the one we choose to employ here is the following. We can split the background charge into two pieces and introduce the $(\frac{1}{2},-\frac{1}{2})$ spectral-flow operator[6]

$$\mu(z,\bar{z}) = e^{-\frac{i}{2}\sqrt{\frac{c}{3}}(X'_R(z)-X'_R(\bar{z}))} \tag{4.9}$$

[5]We will not consider correlation functions involving the observables $\oint dz\ \mathcal{O}^{(1,0)}_\alpha$ or $\oint d\bar{z}\ \mathcal{O}^{(0,1)}_\alpha$ in this paper.

[6]The opposite signs for the left- and right-movers here have to do with the particulars of the A-model topological twist described in section 2.

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at two distinct points, say ξ_1 and ξ_2. Then, we can take ξ_1 and ξ_2 respectively to coincide with the insertions of the first two vertex operators $\mathcal{O}^{(0)}_{\alpha_1}$ and $\mathcal{O}^{(0)}_{\alpha_2}$ and this effectively converts them into vertex operators in the R-sector (see, for example, (2.16)).[7] For simplicity we may set $\xi_1 = 0$, $\xi_2 = \xi$ and later send $\xi \to \infty$. At the end, we can re-express the N-point function (4.8) in terms of the following correlation function in the untwisted $\mathcal{N} = 2$ theory

$$\begin{aligned}\mathcal{F}_{\alpha_1,\dots,\alpha_N} &= \\ &= \lim_{\xi\to\infty} |\xi|^{A_{123}} \left\langle \mathcal{O}^{(0)R}_{\alpha_1}(0)\mathcal{O}^{(0)R}_{\alpha_2}(\xi,\bar\xi)\mathcal{O}^{(0)}_{\alpha_3}(1)\prod_{\ell=4}^{N}\int d^2z_\ell |z_\ell|^{q_{\alpha_\ell}-1}|z_\ell-\xi|^{q_{\alpha_\ell}-1}\mathcal{O}^{(1,1)}_{\alpha_\ell}(z_\ell,\bar z_\ell)\right\rangle ,\end{aligned} \tag{4.10}$$

where q_{α_ℓ} denotes the $\mathrm{U}(1)_R$ charge of the vertex operators $\mathcal{O}^{(0)}_{\alpha_\ell}$ and the constant A_{123} is

$$A_{123} = q_{\alpha_1} + q_{\alpha_2} + q_{\alpha_3} - \frac{c}{6}\,. \tag{4.11}$$

The extra insertions $|z|^{q_{\alpha_\ell}-1}|z-\xi|^{q_{\alpha_\ell}-1}$ and the factor $|\xi|^{A_{123}}$ are needed in order to isolate the dimensionless part of the N-point function in the untwisted theory, which is precisely the one that coincides with the topological string amplitude. For more details on this prescription we refer the readers to [5, 37].

4.2 Computing N-point functions

We now focus on the topologically twisted coset CFT $\mathrm{SL}(2,\mathbb{R})_n/\,\mathrm{U}(1)$ at level n. We want to compute the scattering amplitude of N vertex operators of the form (2.18). According to the general prescription (4.10) in the *untwisted* coset theory we should compute the amplitude

$$\begin{aligned}\mathcal{F}^n_V(\{s_i\};N) &= \\ &= \lim_{\xi\to\infty} |\xi|^{A_{123}} \prod_{\ell=4}^{N}\int d^2z_\ell |z_\ell|^{q_{s_\ell}-1}|\xi-z_\ell|^{q_{s_\ell}-1}\left\langle V^R_{s_1}(0)V^R_{s_2}(\xi,\bar\xi)V_{s_3}(1)V^{(1,1)}_{s_\ell}(z_\ell,\bar z_\ell)\right\rangle .\end{aligned} \tag{4.12}$$

In this case, the constant A_{123} is

$$A_{123} = -\frac{3}{2} + 2s_1 + 2s_2 + 2s_3 + \frac{n+2}{n} \tag{4.13}$$

and the $\mathrm{U}(1)_R$ charges

$$q_{s_\ell} = 1 + 2s_\ell - \frac{n-1}{n}\,. \tag{4.14}$$

For quick reference, we also summarize the explicit form of the vertex operators appearing in (4.12)

$$V^R_s = e^{\frac{i}{2}(H+\bar H)}\Phi_{-\frac12-ns,-\frac12-ns,-\frac12-ns}\; e^{-i\sqrt{2n}s(X-\bar X)}\,, \tag{4.15}$$

$$V_s = e^{-i(H+\bar H)}\Phi^{w=1}_{-\frac12-ns,-\frac12-ns,-\frac12-ns}\; e^{i\sqrt{\frac{2}{n}}(-ns+\frac{n-1}{2})(X-\bar X)}\,, \tag{4.16}$$

$$V^{(1,1)}_s = \Phi^{w=1}_{-\frac12-ns,-\frac32-ns,-\frac32-ns}\; e^{i\sqrt{\frac{2}{n}}(-ns+\frac{n-1}{2})(X-\bar X)}\,. \tag{4.17}$$

[7] From now on, we consider the 0-form vertex operators exclusively in the NS-sector (they are *e.g.* of the form (2.18)).

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In all cases the quantum number s is a half-integer. Also, notice that in (4.17) we have defined the normalization of the (1,1)-form vertex operators to be such that[8]

$$V_s^{(1,1)} = \frac{n}{2(1+2ns)^2}\, G^-_{-1/2}\bar{G}^+_{-1/2} \cdot V_s \,. \tag{4.18}$$

This is needed in order to obtain a correlation function $\mathcal{F}^n_V(\{s_i\};N)$ that is fully symmetric under the interchange of the labels s_i.

For starters, let us consider the selection rules in (4.12) coming from the different U(1)'s. The H-momentum conservation condition is automatically satisfied and the conservation of the X-momentum gives

$$n\sum_{i=1}^{N} s_i = (N-2)\frac{n-1}{2}\,. \tag{4.19}$$

This implies that the J^3-momenta satisfy the equations

$$\sum_{i=1}^{N} m_i + (N-2)\frac{n+2}{2} = \sum_{i=1}^{N} \bar{m}_i + (N-2)\frac{n+2}{2} = 0\,. \tag{4.20}$$

It is now straightforward to compute the correlation function (4.12) with the use of the SRT map (3.6). The U(1) amplitudes are trivial and the $\mathrm{SL}(2,\mathbb{R})$ correlation functions can be recast as correlation functions in Liouville field theory. There are certain things to notice in this computation. First, in order to use the SRT map we must perform an analytic continuation to the quantum numbers of the discrete representations appearing in the topological string. Secondly, notice that the amplitude (4.12) involves $N-2$ $\mathrm{SL}(2,\mathbb{R})$ vertex operators, each one having $w=1$. Hence, the amplitude (4.12) reduces to an $\mathrm{SL}(2,\mathbb{R})$ N-point function with maximal spectral-flow number violation. With the use of the SRT map (3.6) this amplitude will be recast in terms of an N-point function in Liouville field theory. Finally, notice that the J^3 selection rules appearing in (3.6) are precisely the ones derived from the conservation of the X-momentum (4.20). Then, with a small amount of algebra we find that

$$\begin{aligned}
\mathcal{F}^n_V(\{s_i\};N) = {} & (-)^{N+1}\Gamma(0)^N\,\frac{2\pi^{3-2N}c_{n+2}^{N-2}}{\sqrt{n}}\;\delta\Bigg(n\sum_{i=1}^{N} s_i - (N-2)\frac{n-1}{2}\Bigg)\times \\
& \times \lim_{\xi\to\infty} |\xi|^{4+\frac{4}{n}(-ns_2+\frac{n-1}{2})(-ns_1-ns_3+n-1)}\times \\
& \times \prod_{\ell=4}^{N}\int d^2 z_\ell\ |z_\ell|^{\frac{4}{n}(-ns_1+\frac{n-1}{2})(-ns_\ell+\frac{n-1}{2})}\times \\
& \times |1-z_\ell|^{\frac{4}{n}(-ns_3+\frac{n-1}{2})(-ns_\ell+\frac{n-1}{2})}|\xi-z_\ell|^{\frac{4}{n}(-ns_2+\frac{n-1}{2})(-ns_\ell+\frac{n-1}{2})}\times \\
& \times \Big\langle \mathcal{V}_{\alpha_1}(0)\mathcal{V}_{\alpha_2}(\xi,\bar{\xi})\mathcal{V}_{\alpha_3}(1)\mathcal{V}_{\alpha_\ell}(z_\ell,\bar{z}_\ell)\Big\rangle\,,
\end{aligned} \tag{4.21}$$

[8] In the right-moving sector $\bar{G}^+ = \sqrt{\frac{2}{n}}\bar{\psi}\bar{J}^-$ and $\bar{G}^- = \sqrt{\frac{2}{n}}\bar{\psi}^\dagger\bar{J}^+$.

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where the indices α_{s_i} (for $i = 1, 2, \ldots, N$) are given by eq. (2.25) and the Liouville field theory has linear dilaton slope $Q = \sqrt{n} + \frac{1}{\sqrt{n}}$. Already at this stage, simply from the conservation laws and the value of the linear dilaton slope Q we can see quite clearly the emergence of the expected non-minimal bosonic string.

Indeed, if the topologically twisted coset $\mathrm{SL}(2,\mathbb{R})_n/\,\mathrm{U}(1)$ is equivalent to the non-minimal bosonic string of section 2 with $b = \frac{1}{\sqrt{n}}$, then we should be able to reproduce the above computation in bosonic string theory. In other words, we should be able to show that there is agreement between (4.21) and the N-point function

$$\mathcal{F}^{c\leq 1}_{\mathcal{Y}}(\{s_i\}; N) = \lim_{\xi\to\infty} \prod_{\ell=4}^{N} \int d^2 z_\ell \left\langle c\bar{c}\; \mathcal{Y}_{s_1}(0) c\bar{c}\; \mathcal{Y}_{s_2}(\xi,\bar{\xi}) c\bar{c}\; \mathcal{Y}_{s_3}(1) \mathcal{Y}_{s_\ell}(z_\ell,\bar{z}_\ell) \right\rangle . \tag{4.22}$$

Straightforward computation of the U(1) parts of this amplitude yields the following relation

$$\boxed{\mathcal{F}^{n}_{V}(\{s_i\}; N) = (-)^{N+1}\Gamma(0)^N \; \frac{2\pi^{3-2N} c_{n+2}^{N-2}}{\sqrt{n}} \; \mathcal{F}^{c\leq 1}_{\mathcal{Y}}(\{s_i\}; N)\,.} \tag{4.23}$$

This formula is the main result of this paper. We find that the topological string and bosonic string computations agree up to an n dependent diverging factor. Rescaling the topological string vertex operators with the extra factor $-\frac{c_{n+2}}{\pi^2}\Gamma(0)$, *i.e.* sending

$$V_s \to -\Gamma(0)\frac{c_{n+2}}{\pi^2} V_s \tag{4.24}$$

we conclude that the topological string and bosonic string amplitudes agree up to the factor

$$-\frac{2\pi^3}{\sqrt{n}\; c_{n+2}^2}\,. \tag{4.25}$$

We would need to know explicitly the possibly diverging constant c_{n+2} in order to determine this factor. Agreement up to a diverging constant factor was also noted for the three-point functions in [23].

5. Conclusions

In this paper we saw that on the level of correlation functions on the sphere the SRT map offers a very natural and straightforward way to go from the topologically twisted $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset to a corresponding non-critical bosonic string. It is an interesting question whether we can make a similar application of the SRT map to check the bosonic/topological correspondence on the level of more correlation functions involving, for example, tachyons with general $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ labels. But with this point aside, it is already interesting that the set of correlation functions in (4.23) provides us with a sufficiently large set of data to reconstruct the correspondence of [23]. To put it differently, if we did not know about the non-minimal bosonic string of [31] and its relation to the topologically twisted coset, we could have discovered it simply by using the SRT map. This suggests that by using the SRT map perhaps it will be possible to extend the bosonic/topological equivalence to many more situations where the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model is involved. A partial list of interesting problems in this direction is the following.

One possibility is to extend the equivalence of [23] for levels n, which are not integers. Rational values of n are perhaps the most natural and physically interesting extensions. In this case, most of the subtleties have to do with the U(1) part of the non-minimal string and the SRT map is not expected to play a crucial role in this extension.

Another wide class of interesting examples involving the supersymmetric coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ are those that are related to the $\mathcal{N}=2$ topological string in the vicinity of general CY threefold singularities. According to [38] these theories are dual (in a double-scaling limit) to a topological string with target space of the general form $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times$ (some $\mathcal{N}=2$ CFT) and are relevant for the topological sector of corresponding four-dimensional Little String Theories (LST's) [39, 40]. It would be interesting to uncover equivalent non-critical bosonic string theories in these situations and see how they fit into the general picture that anticipates these bosonic theories as deformations of the $c=1$ string [22]. It would also be interesting to see if it is possible to connect this story with the matrix model statements of [15].

An obvious question is whether such bosonic/topological string equivalences are particular to the $\mathcal{N}=2$ topological string, or whether more general string/string equivalences appear also for other topological strings like the $\mathcal{N}=2$ string [41, 42]. It would certainly be worthwhile considering the $\mathcal{N}=2$ string on spaces of the form $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times$(some $\mathcal{N}=2$ CFT) [43, 44], which by the arguments of [38] are dual to $\mathcal{N}=2$ strings in K3 singularities, and compute scattering amplitudes using the SRT map.[9] This computation would be relevant to the physics of the topological sector of six-dimensional LST's. The above issues are currently under investigation [46].

A. $\mathcal{N}=2$ and $\mathrm{SL}(2,\mathbb{R})$ spectral flow conventions

For quick reference we summarize in this appendix the definition of the spectral flow transformation for the $\mathcal{N}=2$ superconformal algebra and the bosonic $\mathrm{SL}(2,\mathbb{R})$ WZW model at level k.

For the $\mathcal{N}=2$ superconformal algebra with central charge c the spectral flow transformation by an amount θ (or $(\theta,\bar\theta)$ if we choose to include both the left- and right-movers) is the $\mathcal{N}=2$ superconformal algebra transformation

$$\begin{aligned}\tilde L_n &= L_n+\theta J_n+\theta^2\frac{c}{6}\delta_{n,0}\\ \tilde G^{\pm}_n &= G^{\pm}_{n\pm\theta}\\ \tilde J_n &= J_n+\frac{c}{3}\theta\,.\end{aligned}\tag{A.1}$$

On the level of primary fields this transformation can be achieved by multiplying with the vertex operator $e^{-i\theta\sqrt{\frac{c}{3}}X_R}$, where X_R is a canonically normalized boson that bosonizes the $\mathrm{U}(1)_R$ current.

[9] For a recent discussion of scattering amplitudes in non-critical $\mathcal{N}=2$ strings see [45].

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In a similar fashion, in $\mathrm{SL}(2,\mathbb{R})$ at level k the spectral flow operation with winding number $w \in \mathbb{Z}$ is defined by the current algebra automorphism[10]

$$\tilde{J}^3_n = J^3_n - \frac{k}{2} w \delta_{n,0}\,, \quad \tilde{J}^\pm_n = J^\pm_{n \mp w}\,. \tag{A.2}$$

On the level of Virasoro generators the spectral flow transformation acts by the shift

$$\tilde{L}_n = L_n - w J^3_n - \frac{k}{4} w^2 \delta_{n,0}\,. \tag{A.3}$$

On vertex operators it corresponds to multiplication with the operator $e^{w\sqrt{\frac{k}{2}}X_3}$. X_3 is the canonically normalized boson that bosonizes the current J^3.

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